Math 256 Exam 3 Summer 2012

Take-home

Name ______________ Date ______________

Please attach ALL of your work if full or partial credit is desired. Circle your final answers. State answers exactly unless otherwise noted. Enjoy!

1. Find a Fourier series for: $f(t) = \begin{cases} \text{depends}, & -\pi < t < 0 \\ t^2, & 0 < t < \pi \end{cases}$

   (a) where $f(t)$ is an even-symmetric function.

   (b) where $f(t)$ is an odd-symmetric function.

2. $\ddot{x} + \alpha^2 x = f(t)$, where $f(t)$ has the Fourier series

$$\sin(\pi t) - \frac{\sin(2\pi t)}{2} + \frac{\sin(3\pi t)}{3} - \frac{\sin(4\pi t)}{4} + \cdots$$

   (a) Find the Fourier series of a periodic solution to this ODE.

   (b) For what values of α is there no periodic solution.

3. $f(t) = t \cdot u(t+1) - t \cdot u(t-2)$

   (a) Sketch the graph of $f(t)$.

   (b) Sketch the graph of the generalized derivative, $f'(t)$.

   (c) Write formula for the generalized derivative, $f'(t)$

4. Find the $\mathscr{L}$aplace transforms for the following functions:

   (a) $e^{-t}\sin(3t)$

   (b) $e^{2t}(t^2 - 3t + 2)$

5. Find the following inverse $\mathscr{L}$aplace tranforms

   (a) $\mathscr{L}^{-1}\left[\frac{1}{s^2 - 2}\right]$

   (b) $\mathscr{L}^{-1}\left[\frac{1}{s^2 - s - 6}\right]$

   (c) $\mathscr{L}^{-1}\left[\frac{s + 10}{s^3 + 2s^2 + 10s}\right]$

6. Use convolution to find the inverse $\mathscr{L}$aplace transform of $\frac{1}{(s^2+4)^2}$

7. Solve the following Initial Value Problems

   (a) $\ddot{x} + 4\dot{x} + 4x = e^{-2t}$, $x(0) = 0$, $\dot{x}(0) = 0$

   (b) $\ddot{x} + 3\dot{x} + 2x = t$, $x(0) = 1$, $\dot{x}(0) = -1$

(c) $\ddot{x} + 2\dot{x} + 2x = \sin(t)$, $x(0) = 0$, $\dot{x}(0) = 1$

(d) $\ddot{x} - 4\dot{x} + 5x = e^{2t}\cos(t)$, $x(0) = 0$, $\dot{x}(0) = 0$

(e) $\ddot{w} + 2\dot{w} + 3w = \delta(t)$, $w(0) = 0$, $\dot{w}(0) = 1$

8. Let $p(D)$ be the operator whose unit impulse response is given by $w(t) = e^{-3t} - e^{-4t}$

   (a) Using convolution, find the unit step response of this operator: the solution to $p(D)v = u(t)$ with initial rest conditions, i.e. $v(0) = 0 = \dot{v}(0)$

   (b) What is the characteristic polynomial, $p(s)$.

9. Solve the system in two ways

$$\begin{aligned} \dot{x} &= x + y \\ \dot{y} &= y \end{aligned} \qquad\qquad \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

10. Solve the following systems using matrix methods:

   (a)
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

   (b)
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

   (c)
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

   (d)
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

11. Convert each second degree ODE to a system using $y = \dot{x}$, then write each system in matrix form, solve the system, and make a phase portrait:

   (a) $\ddot{x} + 4\dot{x} + 3x = 0$

   (b) $\ddot{x} + \dot{x} + \frac{5}{2}x = 0$